Permutations preserving the convergence or the sum of series – a survey

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Abstract. Presented paper is a survey of many authors' achievements in the research subject matter concerning the permutations preserving the convergence or the sum of series and the algebraic properties of the families of such permutations. The convergence classes of divergent permutations will be also discussed. This survey has been treated widely, however not exhaustive.

Keywords: convergent permutation, divergent permutation, convergence class.

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1. Introduction

In classical Riemann Derangement Theorem about rearrangement of the conditionally convergent series the series is fixed and the rearranged series vary. In case when we fix the permutation p on $\mathbb{N}$ and we variate the selection of conditionally convergent series a number of new problems appear, including the problem concerning the form of sets of limit points of the series rearranged by p, which is a dual issue to the problem described in the above Riemann Derangement Theorem. This new problem was solved by Nash-Williams and White [37] just in 1999. Simultaneously we note that many special cases of this problem were discovered earlier [29, 66, 63].

Some other problems turned out to be interesting and essential as well, for example the problem of combinatoric description of the convergent (divergent) permutations or the permutations preserving the sum. Discussion of such problems is the object of this paper, frame of which was given by my habilitation thesis [72].

Remark 1.1. This survey does not contain any constructions of permutations on $\mathbb{N}$ distinguished and discussed in this paper, which can make the considered issues quite

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rough for the Readers not initiated into the subject matter. Remedy for this technical problem can be the series of papers, perfectly supplementing this deficiency, which I recommend to all the interested Readers (see [44, 52, 53, 60, 73, 79]).

2. Basic ideas and distinguished sets of permutations

Family of all permutations on $\mathbb{N} = \{1, 2, 3, ...\}$, it means the bijections of the set of natural numbers on itself, will be denoted by $\mathfrak{P}$. Permutation $p \in \mathfrak{P}$ preserving the convergence of all rearranged by p convergent series of real terms will be called the convergent permutation. In other words, permutation $p \in \mathfrak{P}$ is convergent if for each convergent series $\sum a_n$ of real terms, the series $\sum a_{p(n)}$ rearranged by permutation p is convergent as well. Family of all convergent permutations will be denoted by symbol $\mathfrak{C}$.

For the contrast, permutations belonging to set $\mathfrak{D} := \mathfrak{P} \setminus \mathfrak{C}$ will be called the divergent permutations. Thus, permutation $p \in \mathfrak{P}$ is the divergent permutation if there exists a convergent series $\sum a_n$ of real terms which is rearranged by permutation p into a divergent series $\sum a_{p(n)}$. Let us also introduce, after Kronrod [34], the following families of permutations (I have distinguished these families independently in 90's, without knowing the Kronrod's work):

- $\mathfrak{CC}$ and $\mathfrak{CD}$ are the subfamilies of $\mathfrak{C}$ composed of permutations p called the two-sided or one-sided convergent permutations in dependence on that, whether the inverse permutation p^{-1} is convergent or divergent, respectively,
- $\mathfrak{DC}$ and $\mathfrak{DD}$ are, by analogy to $\mathfrak{CC}$ and $\mathfrak{CD}$, the subfamilies of $\mathfrak{D}$ of permutations p called the two-sided or one-sided divergent permutations in dependence on that, whether permutation p^{-1} is convergent or divergent.

Moreover, we indicate permutations $p \in \mathfrak{P}$ preserving the sum, it means satisfying the condition: for each convergent real series $\sum a_n$ if the series $\sum a_{p(n)}$ rearranged by p is convergent, then it preserves the sum, that is $\sum a_{p(n)} = \sum a_n$. Henceforward we will denote by the same symbol $\sum a_n$ the series, it means the appropriate sequence of partial sums as well as its sum, if only the given series is convergent. The proper interpretation will depend on the context of discussion. Family of permutations preserving the sum will be denoted by symbol $\mathfrak{S}$. Permutations belonging to set $\mathfrak{I} := \mathfrak{P} \setminus \mathfrak{S}$ will be called the substantially singular permutations. Thus, each substantially singular permutation $p \in \mathfrak{P}$ rearranges some convergent series $\sum a_n$ of real terms into series $\sum a_{p(n)}$ convergent as well, but of changed value of the sum, it means $\sum a_{p(n)} \neq \sum a_n$.

We distinguish also some important for further discussion subfamily $\mathfrak{F}$ of family $\mathfrak{S}$. We say that permutation $p \in \mathfrak{P}$ belongs to $\mathfrak{F}$, if there exists a finite partition $N_1, N_2, \ldots, N_{n(p)}$ of the set of natural numbers such that the restrictions $p\big|_{N_i}$, $i = 1, 2, \ldots, n(p)$, are the increasing maps. One more auxiliary idea will be essential for further discussion. We say that the finite and nonempty set $A \subset \mathbb{N}$ is a union of k $\underline{\underline{\text{m}}}$utually $\underline{\underline{\text{s}}}$eparated $\underline{\underline{\text{i}}}$ntervals, in short: k msi, if there exists partition $I_1, I_2, \ldots, I_k$ of set A composed from k intervals (segments) of natural numbers, it means from the sets of successive natural numbers such that $\operatorname{dist}(I_i, I_j) \geqslant 2$ for any $i, j = 1, 2, \ldots, k$, $i \neq j$. For convenience we will also say the given set $A \subset \mathbb{N}$ or a few of sets $A_1, \ldots, A_r \subset \mathbb{N}$

are the unions of at most n msi for some $n \in \mathbb{N}$, if each of them is a union of $k(A)$ msi or, respectively, of $k(A_1)$ msi, $k(A_2)$ msi,..., $k(A_r)$ msi and all these numbers are not greater than n. Number n plays here a role of majorant of the given set of numbers. We say that the given sequence $\{A_n\}$ of finite and nonempty subsets of $\mathbb{N}$ is increasing if $A_n < A_{n+1}$ for every $n \in \mathbb{N}$ and we write $A < B$ if $a < b$ for any $a \in A$, $b \in B$, where $A, B \subset \mathbb{N}$ are the nonempty sets. Notation $A \subset B$ will be used only in case of strict inclusion.

3. Selected algebraic properties of the distinguished sets of permutations

We say that family $\mathfrak{A} \subset \mathfrak{P}$ is algebraically small if $\mathfrak{P} \setminus G(\mathfrak{A}) \neq \varnothing$, where $G(\mathfrak{A})$ is a group of permutations generated by $\mathfrak{A}$. Next, we say that family $\mathfrak{A} \subset \mathfrak{P}$ is algebraically big if $\mathfrak{A} \circ \mathfrak{A} = \mathfrak{P}$, where operation $\circ$ of composition of two nonempty sets of permutations $\mathfrak{A}$ and $\mathfrak{B}$ is defined as follows

$$\mathfrak{A} \circ \mathfrak{B} := \{p \circ q : p \in \mathfrak{A} \text{ and } q \in \mathfrak{B}\},$$

whereas $p \circ q(n) := p(q(n))$, $n \in \mathbb{N}$. Families $\mathfrak{A} \subset \mathfrak{P}$ which are neither algebraically small nor algebraically big will be called as the sets algebraically medial. There exist the sets of generators of $\mathfrak{P}$ which are the sets algebraically medial [70]. Let us also note that

- family $\mathfrak{C}$ is algebraically small (Pleasants [43, 44]),
- families $\mathfrak{S}$ and $\mathfrak{I}$ are algebraically big (Kronrod [34] Pleasants [43], Wituła [68], in fact we know many algebraically big subsets of family $\mathfrak{S}$ – some of them will be presented in the further parts of this survey),
- families $\mathfrak{DC}$ i $\mathfrak{CD}$ are semigroups (Wituła [65, 74]), more precisely we have $\Phi \circ \Phi = \Phi$ for each $\Phi \in \{\mathfrak{DC}, \mathfrak{CD}, \mathfrak{C}, \mathfrak{C}^{-1}\}$,
- family $\mathfrak{DD}$ is algebraically big (Wituła [65]),
- I proved that (see [65, 74]):

$$\mathfrak{DC} \circ \mathfrak{DD} = \mathfrak{DD} \circ \mathfrak{DC} = \mathfrak{DC} \cup \mathfrak{DD} = \mathfrak{D}$$

and

$$\mathfrak{CD} \circ \mathfrak{DD} = \mathfrak{DD} \circ \mathfrak{CD} = \mathfrak{CD} \cup \mathfrak{DD} = \mathfrak{D}^{-1}.$$

Also the following equalities[1] $\mathfrak{DD}^k \circ \mathfrak{DD}^l = \mathfrak{P}$ are satisfied for any $k, l \in \mathbb{N}$, $k, l \geqslant 2$ (Wituła [70]), where $\mathfrak{A}^k := \{p^k : p \in \mathfrak{A}\}$ for any nonempty $\mathfrak{A} \subset \mathfrak{P}$ and p^k denotes the k-fold composition of p with itself (thus, symbol of type $\mathfrak{DD}^k$ will denote the kth power of set $\mathfrak{DD}$, it means $(\mathfrak{DD})^k$). Moreover we have $\mathfrak{DD} \setminus \bigcup_{k=2}^{\infty} \mathfrak{DD}^k \neq \varnothing$, but it is unknown whether the equality $\bigcup_{k=1}^{\infty} \mathfrak{DD}^k = \mathfrak{P}$ holds. Family $\mathfrak{CC}$ is a group with regard

[1] **Remark.** Identified here the algebraically big sets $\mathfrak{S}$ and $\mathfrak{DD}$ I can decompose into countably many algebraically big subsets. However I do not know whether it is true for each algebraically big subset of family $\mathfrak{P}$.

to composition of mappings and plays the role of unity with regard to operation $\circ$ for many from among sets discussed here. More precisely, we have [68, 74]:

$$\mathfrak{CC} \circ \mathfrak{A} = \mathfrak{A} \circ \mathfrak{CC} = \mathfrak{A}$$

for $\mathfrak{A} \in \{\mathfrak{CC}, \mathfrak{C}, \mathfrak{C}^{-1}, \mathfrak{CD}, \mathfrak{D}, \mathfrak{DC}, \mathfrak{DD}, \mathfrak{S}_0\}$ (definition of family $\mathfrak{S}_0$ will be given on page 175). One can easily verify that family $\mathfrak{CC}$ is the maximal, with regard to inclusion, group of permutations included in $\mathfrak{C}$. Next, from the fact that $\mathfrak{CD}$ is the semigroup, it results in particular that set $\mathfrak{CC}$ contains all the torsion elements of semigroup $\mathfrak{C}$. The family τ_C of torsion elements of this semigroup is a normal quotient of group $\mathfrak{CC}$ and, which is more, of the infinite index (more precisely, index $|\mathfrak{CC} : \tau_C| = \mathfrak{c}$). We have similar result for the subgroup τ of torsion elements of group $\mathfrak{P}$. Certainly $\tau_C \subset \tau$ and $\tau \cap \mathfrak{DD} \neq \varnothing$. Furthermore, we have here a number of unsolved problems like, for instance, whether τ_C (τ respectively) is the maximal, with regard to inclusion, normal subgroup in $\mathfrak{CC}$ (in $\mathfrak{P}$ respectively).

E.H. Johnston [31] introduced interesting subgroup $\mathfrak{R}$ of group $\mathfrak{CC}$, composed from permutations $p \in \mathfrak{P}$ satisfying condition: $\sup\{\mathrm{card}(I \setminus p(I)) : I \subset \mathbb{N} \text{ is an interval}\} < \infty$.[2] One can show that $\mathfrak{R}$ is not the normal subgroup of group $\mathfrak{CC}$ (Johnston proved this fact only for group $\mathfrak{P}$). One can also prove that $\mathfrak{CC}$ is not the normal subgroup in semigroup $\mathfrak{C}$ (it means, there exists $\tau \in \mathfrak{C}$ such that $\tau\mathfrak{CC}\tau^{-1} \cap \mathfrak{D} \neq \varnothing$) – G.S. Stoller [56]. I have generalized this fact importantly by proving relations (see [71]):

$$(p\mathfrak{CC}q) \cap \mathfrak{DD} \neq \varnothing \quad \text{and} \quad (q\mathfrak{CC}p) \cap \mathfrak{DD} \neq \varnothing$$

for any $p \in (\mathfrak{CD} \cup \mathfrak{DD})$ and $q \in \mathfrak{D}$. Additionally, I have proven that for any permutation $p \in \mathfrak{P}$ we have

$$p\mathfrak{D}p^{-1} \subseteq \mathfrak{D} \Leftrightarrow p \in \mathfrak{CC} \quad \text{and} \quad p\mathfrak{DD}p^{-1} \subseteq \mathfrak{DD} \Leftrightarrow p \in \mathfrak{CC},$$

where if $p \in \mathfrak{CC}$ then the above inclusions turn into equalities (see [71, Th. 2.7]).[3]

4. Selected characterizations of convergent and divergent permutations

A number of various characterizations of convergent, divergent and other permutations are known – see among others [22, 26, 35, 43, 49, 51, 52, 53, 57]. Let me present few of them:

- $p \in \mathfrak{C}$ if and only if there exists constant $k = k(p) \in \mathbb{N}$ such that the set $p([1, n] \cap \mathbb{N})$ is a union of at most k msi for each $n \in \mathbb{N}$ (in other words, if

[2] **Remark.** Różański et al. [46] have completed the description of elements of subgroup $\mathfrak{R}$ on the basis of the following equivalence relations ϱ defined on family b_+ of the bounded sequences of positive real numbers: $\{a_n\}\varrho\{b_n\} \Leftrightarrow$ for every increasing sequence of positive integers $\{n_k\}$ the series $\sum a_{n_k}$ and $\sum b_{n_k}$ are simultaneously convergent or divergent.

[3] **Remark.** Existence of divergent permutations (two-sided divergent permutations, respectively) in sets $\tau\mathfrak{R}\tau^{-1}$, $\tau \in \mathfrak{C}$, $p\mathfrak{R}q$, $p \in \mathfrak{CD}$ and $q \in \mathfrak{DC}$, remains problematic, where $\mathfrak{R}$ is the group introduced by Johnston.

<table>
<tr><td rowspan="4">$\mathfrak{P}$</td><td rowspan="2">$\mathfrak{C}$</td><td>$\mathfrak{CC}$
$\mathfrak{CC} \cap \mathfrak{F}$</td><td rowspan="3">$\mathfrak{S}$</td><td rowspan="3">$\mathfrak{S}_\circ$
$\mathfrak{D}(1)$
$\mathfrak{S}_\circ \setminus \mathfrak{G}$
$\mathfrak{D}_2$
$\mathfrak{D}(1) \setminus \mathfrak{G}$</td><td rowspan="3">$\mathfrak{G}$
$\mathfrak{C}_2 \cup \mathfrak{D}_2$
$\mathfrak{F} \subset \mathfrak{C}_2 \Rightarrow \mathfrak{DD} \cap \mathfrak{G} \neq \emptyset$
$\mathfrak{F} \setminus \mathfrak{D}_2 \neq \emptyset \Rightarrow \mathfrak{C}_2 \setminus \mathfrak{D}_2 \neq \emptyset$</td></tr>
<tr><td>$\mathfrak{CD}$
$\mathfrak{CD} \cap \mathfrak{F}$</td></tr>
<tr><td rowspan="2">$\mathfrak{D}$</td><td>$\mathfrak{DC}$
$\mathfrak{D}(1) \cap \mathfrak{DC}$
$\mathfrak{F} \cap \mathfrak{DC}$</td></tr>
<tr><td>$\mathfrak{DD}$
$\mathfrak{D}(1) \cap \mathfrak{DD}$
$\mathfrak{F} \cap \mathfrak{DD}$</td><td>$\mathfrak{I}$
$\mathfrak{F} \cap \mathfrak{I}$</td><td></td><td></td></tr>
</table>

All sets displayed in the table are nonempty. In some blocks with the aid of smaller types there are indicated selected subsets of sets presented at the top of these blocks. Case when the vertical side of some block is included in the vertical side of another block denotes the appropriate inclusion between sets displayed at the top of these blocks. Inclusions $\mathfrak{S}_\circ \subset \mathfrak{G}$ and $\mathfrak{S} \subset \mathfrak{G}$ are the hypotheses formulated by me.

$\limsup_{n\to\infty} t(p,n) \leqslant k$, that is, when $\limsup_{n\to\infty} t(p,n) < \infty$, where $t(p,n)$ denotes the number of msi partitioning set $p([1,n])$), R.P. Agnew (1955) [2], N. Bourbaki (1951) [9],

- $p \in \mathfrak{D} \Leftrightarrow \limsup_{n\to\infty} t(p,n) = \infty$ (characterization dual to the previous one),
- $p \in \mathfrak{D}$ if and only if for any $r,s \in \mathbb{N}$ there exists the increasing sequence of natural numbers $\{x_n\}_{n=1}^{2r}$ spliced by p, such that $x_1 > s$ and each of two following sequences $\{p(x_n)\}_{n=1}^{r}$ and $\{p(x_{n+r})\}_{n=1}^{r}$ is monotonic (one can also assume that $\min\{p(x_n)\}_{n=1}^{2r} > s$), Wituła (1995) [80, 66].

Remark 4.1. An increasing sequence $\{x_n\}_{n=1}^{2r}$ of positive integers is spliced by p if the increasing sequences composed from elements of sets $\{p(x_n) : n = 1,\dots,r\}$ and $\{p(x_{n+r}) : n = 1,\dots,r\}$ are alternating. We say that increasing sequences of natural numbers: $y_1, y_2, \dots, y_r$ and $z_1, z_2, \dots, z_r$ are alternating if $y_1 < z_1 < y_2 < z_2 < \dots < y_r < z_r$ or $z_1 < y_1 < z_2 < y_2 < \dots < z_r < y_r$.

It should be emphasized that the idea of splicing is connected with any divergent permutation which is justified by sequences, assigned to these permutations, conglomerating intervals and growing the numbers of intervals (see Section 7). Moreover, by using properties of these sequences one can prove the characterization of divergent permutation presented in here (see Remark 7.7), in the way alternative to proofs from papers [80, 66].

- $p \in \mathfrak{C}$ if and only if there exists $r \in \mathbb{N}$ such that for each increasing sequence $\{x_n\}_{n=1}^{2N}$ of natural numbers spliced by p the inequality $N \leqslant r$ holds (this characterization is dual to the previous characterization restricted only to the condition of splicing the sequence $\{x_n\}_{n=1}^{2r}$ by p).

First of the above four characterizations introduced by Bourbaki and Agnew has been generalized by me into functions $f : \mathbb{N} \to \mathbb{N}$ preserving convergence of the convergent real series rearranged by them[4] (see [64]). Functions of that kind will be shortly called the convergent functions and their description, discovered by me, is the following.

Function $f:\mathbb{N}\to\mathbb{N}$ is the convergent function if and only if there exists a natural number $t=t(f)$ such that for each interval $I\subset\mathbb{N}$ there exists a partition $\mathfrak{I}_1,\mathfrak{I}_2,\ldots,\mathfrak{I}_s$ of interval I possessing the following properties:

1) $s\leqslant t$,
2) all restrictions $f\big|_{\mathfrak{I}_i}$, $i=1,2,\ldots,s$, are injections,
3) each of sets $f(\mathfrak{I}_i)$, $i=1,2,\ldots,s$, is a union of at most t msi.

Remark 4.2. The above description implies that the convergent functions, and in particular the convergent permutations, with respect to the series in any normed space (and even in any linear-topological space) are functions (permutations, respectively) **preserving the Cauchy condition**. It is important, among others, since the convergent permutations preserve also the sums of convergent series rearranged by them, whereas the convergent functions, which are not the permutations of the set of natural numbers, do not possess this property any more!

Remark 4.3. Characterization of the convergent functions, introduced by me, and the above remark can be transferred without changes onto the vector series in the normed spaces, as well as generally in the linear-topological spaces, over the fields of characteristic zero.

Remark 4.4. Professor Lech Drewnowski in paper [18, Lemmas 4.4, 4.5 and Prop. 4.6] discusses many equivalent characterizations of convergent functions and, what is the most essential, he proves that they characterize the functions preserving convergence of the series and, independently, the functions transforming the convergent series (bounded series, respectively) into the bounded series and the functions transforming the Cauchy sequences into the Cauchy sequences, etc. in the linear-topological spaces and even in the normed F-spaces.

Remark 4.5. M.A. Sarigöl [51] has presented the characterization of permutations $p\in\mathfrak{P}$ preserving the property of bounded variation of scalar sequences (these are exactly the permutations, the inverse permutations of which are convergent). Drewnowski in paper [18] has generalized this characterization onto functions $f:\mathbb{N}\to\mathbb{N}$ and in case of permutations also onto the sequences of bounded variation in the linear-topological spaces and in the normed F-spaces. In particular, I have noticed here an intriguing result (not recorded either in paper [51] or in paper [18]): if $p\in\mathfrak{P}$ and $\limsup\limits_{n\to\infty}|p^{-1}(n+1)-p^{-1}(n)|<\infty$, then $p\in\mathfrak{C}$ (see [18, Prop. 6.3]).

5. Family $\mathfrak{S}_0$ of permutations preserving the sum

Till the last year I was convinced that the combinatoric characterization of permutations preserving the sum was still unknown. It was the reason for preparing paper [68] in which I have distinguished the following family $\mathfrak{S}_0$ of permutations

[4] Function $f:\mathbb{N}\to\mathbb{N}$ will be called the function preserving convergence of the (real) series if for each convergent series $\sum a_n$ of real terms the series $\sum a_{f(n)}$ is convergent as well.

on $\mathbb{N}$. We say that permutation $p \in \mathfrak{P}$ belongs to $\mathfrak{S}_0$ if there exists a natural number $k = k(p)$ such that for each $n \in \mathbb{N}$ the nonempty finite sets $A_n, B_n \subset \mathbb{N}$ exist and satisfy the conditions:

1) $p(A_n) = B_n$,
2) $[1, n] \subset A_n$,
3) each of sets A_n and B_n is a union of at most k msi.

By using the Cauchy condition one can easily verify that $\mathfrak{S}_0 \subseteq \mathfrak{S}$ and I formulated the conjecture that the equality $\mathfrak{S}_0 = \mathfrak{S}$ holds, however I could not prove it (in next subsection it will appear that in 1999, and this date is not a mistake, this conjecture was proved true by Nash-Williams and White – see [37]). I based this conjecture on the following reason. All the known till now examples of permutations preserving the sum are the permutations belonging to $\mathfrak{S}_0$. One can find here the really nontrivial constructions, like for example given in paper [29] the construction of permutation $p \in \mathfrak{S}_0$ such that

$$\lim_{n\to\infty} t(p; n) = \lim_{n\to\infty} t(p^{-1}; n) = \infty, \tag{1}$$

where for each $n \in \mathbb{N}$ symbol $t(p; n)$ denotes the number of msi partitioning set $p([1, n])$.

Remark 5.1. Condition (1) is connected with one more interesting characterization, this time of permutations $p \in \mathfrak{P}$ rearranging some convergent real series $\sum a_n$ into series $\sum a_{p(n)}$ divergent to ∞. Hu and Wang in the mentioned paper [29] have proven that the necessary and sufficient condition, under which this fact is happening for given permutation $p \in \mathfrak{P}$, is that $\lim\limits_{n\to\infty} t(p; n) = \infty$.[5]

I have proven the above characterization of Hu-Wang independently in paper [66], and I am convinced that the proof given by me is more intelligible.

In paper [68] I have proven many algebraic and combinatoric properties of family $\mathfrak{S}_0$. Let me begin with the condition guaranteeing that $p \in \mathfrak{S}_0$, namely: $\liminf\limits_{n\to\infty} t(p; n) < \infty$. It is not in the least the necessary condition, which results from the previously given example by Hu-Wang (see condition (1)).

I present now the collection of selected algebraic relations for family $\mathfrak{S}_0$ (on the basis of Theorem 2.4 in paper [68]):

(i) $\mathfrak{S}_0^{-1} = \mathfrak{S}_0$ which is compatible with equality $\mathfrak{S}^{-1} = \mathfrak{S}$
(both equalities result from the definitions of families $\mathfrak{S}_0$ and $\mathfrak{S}$, respectively).

Remark 5.2. If $G \subset \mathfrak{P}$ is a group and $G \neq \mathfrak{P}$ then $(\mathfrak{P} \setminus G)^{-1} = \mathfrak{P} \setminus G$, $G \circ (\mathfrak{P} \setminus G) = (\mathfrak{P} \setminus G) \circ G = \mathfrak{P} \setminus G$, but either $\mathfrak{S}_0$ or $\mathfrak{S}$ are not groups, since both of them are the algebraically big sets.

(ii) $\mathfrak{C} \cup \mathfrak{C}^{-1} \subset \mathfrak{S}_0$ and $\mathfrak{S}_0 \cap \mathfrak{D}\mathfrak{D} \neq \varnothing$
(the comment is required for the second relation resulting easily from inclusion $\mathfrak{D}(1) \subset \mathfrak{S}_0$, where $\mathfrak{D}(1)$ is the family of such permutations $p \in \mathfrak{D}$ for which $p([1, n] \cap \mathbb{N}) = [1, n] \cap \mathbb{N}$ for infinitely many $n \in \mathbb{N}$, it means fulfilling condition

[5] Despite of the appearances, constructing such permutation is not difficult – essential detail of this construction is described in item (v) of Lemma 7.1 in Section 7.

$\liminf\limits_{n\to\infty} t(p,n) = 1$. Family $\mathfrak{D}(1)$ is algebraically big, which is already proven by G.S. Stoller [56]. Additionally I have proven that $\mathfrak{D}(1) \circ \mathfrak{I} = \mathfrak{I} \circ \mathfrak{D}(1) = \mathfrak{P}$ – Theorem 2.7 in paper [68]).

(iii) $\mathfrak{C} \circ \mathfrak{S}_0 = \mathfrak{S}_0 \circ \mathfrak{C}^{-1} = \mathfrak{S}_0$ and $(\mathfrak{CD} \circ \mathfrak{S}_0) \cup (\mathfrak{S}_0 \circ \mathfrak{DC}) \subset \mathfrak{S}_0 \subset (\mathfrak{DC} \circ \mathfrak{S}_0) \cap (\mathfrak{S}_0 \circ \mathfrak{CD})$ (let us recall that family $\mathfrak{CC}$ is unity with regard to operation $\circ$ for many of discussed here families of permutations. Bigger with regard to inclusion families $\mathfrak{C}$ or $\mathfrak{C}^{-1}$, as it can be seen, satisfy the rule of one-sided unities, since we have

$$\mathfrak{C} \circ \mathfrak{S} = \mathfrak{S} \circ \mathfrak{C}^{-1} = \mathfrak{S}, \quad \mathfrak{CD} \circ \mathfrak{C} = \mathfrak{CD}, \quad \mathfrak{C}^{-1} \circ \mathfrak{DC} = \mathfrak{DC},$$

where $\mathfrak{DC} \cup \mathfrak{CD} \subset \big(\mathfrak{C} \circ \mathfrak{CD} \cap \mathfrak{C}^{-1} \circ \mathfrak{DC}\big)$. Moreover, I have proven that $\mathfrak{S} \circ \mathfrak{CC} = \mathfrak{S}$ and $\mathfrak{I} \circ \mathfrak{CC} = \mathfrak{I}$).

(iv) $\mathfrak{CD} \circ \mathfrak{S}_0 \circ \mathfrak{DC} \subset \mathfrak{S}_0 \subset \mathfrak{DC} \circ \mathfrak{S}_0 \circ \mathfrak{CD}$
(similarly like the second one from among relations (iii), also the above relation shows well the subtle set-theoretic difference between the occuring sets, whereas the qualitative nature of these differences needs still to be investigated),

(v) $\mathfrak{D}_2 \subset \mathfrak{S}_0$ and $\mathfrak{C}_2 \cap \mathfrak{I} \neq \varnothing$, which implies that $\mathfrak{D}_3 \cap \mathfrak{I} \neq \emptyset$, where $\mathfrak{C}_1 := \mathfrak{CD}$, $\mathfrak{D}_1 := \mathfrak{DC}$, $\mathfrak{C}_{k+1} := \mathfrak{D}_k \circ \mathfrak{C}$, $\mathfrak{D}_{k+1} := \mathfrak{C}_k \circ \mathfrak{D}$ for each $k \in \mathbb{N}$. It is known that $\mathfrak{C}_k \subseteq \mathfrak{C}_{k+1}$, $\mathfrak{D}_k \subseteq \mathfrak{D}_{k+1}$, $\mathfrak{D}_k \cup \mathfrak{C}_k \subseteq \mathfrak{D}_{k+1} \cap \mathfrak{C}_{k+1}$ for each $k \in \mathbb{N}$ and, what is the most important, $\bigcup\limits_{k\in\mathbb{N}} \mathfrak{C}_k = \bigcup\limits_{k\in\mathbb{N}} \mathfrak{D}_k = \mathfrak{G}$, where, let us recall, $\mathfrak{G}$ is the group of permutations generated by family $\mathfrak{C}$ (see [68]). We already know that $\mathfrak{C}_2 \setminus \mathfrak{D}_2 \neq \varnothing$ (see [68, 71]]), however we do not know whether family $\mathfrak{C}_2 \setminus \mathfrak{S}_0$ is algebraically big. I have proven, independently of Kronrod, that set $\mathfrak{I} = \mathfrak{D} \setminus \mathfrak{S}$ is algebraically big [68, Theorem 2.7]. Still unknown is also the answer to question whether there exists $k \in \mathbb{N}$ such that $\mathfrak{D}_k = \mathfrak{D}_{k+1}$ or $\mathfrak{C}_k = \mathfrak{C}_{k+1}$ (let us note that such equality implies that, respectively, $\mathfrak{D}_k = \mathfrak{D}_{k+l} = \mathfrak{C}_{k+l}$ or $\mathfrak{C}_k = \mathfrak{C}_{k+l} = \mathfrak{D}_{k+l}$ for each $l \in \mathbb{N}$). I have proven as well that family $\mathfrak{P} \setminus \mathfrak{G}$ is algebraically big and $\mathfrak{D}(1) \setminus \mathfrak{G} \neq \emptyset$[6] (see [77]). In particular, it implies that $\{g\} \circ (\mathfrak{P} \setminus \mathfrak{G}) = (\mathfrak{P} \setminus \mathfrak{G}) \circ \{g\} = \mathfrak{P} \setminus \mathfrak{G}$ for any permutation $g \in \mathfrak{G}$. These equalities hold also for any group $G \subset \mathfrak{P}$ (taken in place of group $\mathfrak{G}$), such that $\mathfrak{P} \setminus G$ is algebraically big. In reference to item (i) let us additionally notice that equalities $\mathfrak{C}_{2k}^{-1} = \mathfrak{C}_{2k}$, $\mathfrak{D}_{2k}^{-1} = \mathfrak{D}_{2k}$, $\mathfrak{C}_{2k-1}^{-1} = \mathfrak{D}_{2k-1}$ and $\mathfrak{D}_{2k-1}^{-1} = \mathfrak{C}_{2k}$ hold for every $k \in \mathbb{N}$.

Another important fact should be also emphasized, namely, that **permutations $p \in \mathfrak{S}$ are the only functions $f : \mathbb{N} \to \mathbb{N}$ preserving the sum of series rearranged by them**. In other words, if for the given convergent real series $\sum a_n$ the f-rearranged series $\sum a_{f(n)}$ is convergent as well and $\sum a_{f(n)} = \sum a_n$, then function f is a permutation. Proof of this fact, by contradiction of thesis, can be immediately generalized onto the vector series in any nontrivial normed space (one should discuss separately the spaces over the fields of characteristic zero and over the fields of characteristic different than zero – see [64]).

[6] In paper [71] the example (Example 1) of permutation $p \in \mathfrak{D}(1) \setminus (\mathfrak{C}_2 \cup \mathfrak{D}_2)$ is given, which is the slightly weaker relation. Whereas, it is noticed there (Remark 2.4) that $\mathrm{card}(\mathfrak{D}(1) \setminus (\mathfrak{C}_2 \cup \mathfrak{D}_2)) = \mathfrak{c}$.

Nash-Williams and White's exciting papers. Equality $\mathfrak{S}_0 = \mathfrak{S}$ holds

Last year (more precisely in October 2013), when I was preparing my lecture for presenting my habilitation thesis in the Łódź University, I read again paper [37] by Nash-Williams and White concerning the form of cluster set of the sequence of partial sums of $\sum a_{p(n)}$, i.e. the convergent real series rearranged by a given permutation $p \in \mathfrak{P}$. I noticed almost immediately (which surprised me extremely) that this paper contains the positive solution of my conjecture claiming that relation

$$\mathfrak{S}_0 = \mathfrak{S}$$

is true. Nash-Williams and White proved the following theorem.

Theorem 5.3. *Permutation $p \in \mathfrak{P}$ preserves the sum of rearranged series if and only if the width of p is finite (the width of permutation $p \in \mathfrak{P}$ is a fundamental conception introduced by Nash-Williams and White in paper [37]) or, equivalently, if and only if $p \in \mathfrak{S}_0$.*

We note that a proper part of this fact, i.e. the implication: if the width of p is finite then $p \in \mathfrak{S}_0$ is formulated in Proposition 2.2 of [37]. Let us emphasize that Nash-Williams and White in paper [37] do not recall the definition or even the conception of permutations preserving the sum which was certainly the reason of not noticing, by myself and other readers, the characterization of permutations preserving the sum described in Section 5. We also note that next papers [38, 39], published by the same authors, concern the generalizations of theorems contained in [37] onto the series in finitely dimensional spaces. All three papers [37, 38, 39] made by Nash-Williams and White are, with no doubts, the very important events in the discussed subject matter.

6. Family of permutations decomposable into the finite sum of increasing maps

Subject matter of my research was also the family of permutations $\mathfrak{F} \subset \mathfrak{P}$, original against a background of previously discussed families, composed from permutations $p \in \mathfrak{P}$ for which there exists the finite partition $N_1, N_2, \ldots, N_{n(p)}$ of the set of natural numbers such that $p\big|_{N_i}$ is the increasing map for each $i = 1, 2, \ldots, n(p)$. Definition of family $\mathfrak{F}$ implies also another, dual characterization of permutations belonging to $\mathfrak{F}$. So, permutation $p \in \mathfrak{P}$ belongs to $\mathfrak{F}$ if and only if the value of

$$k(p) := \sup\{\mathrm{card}(A) : \varnothing \neq A \subset \mathbb{N} \text{ and } p\big|_A \text{ is decreasing map}\}$$

is finite (see [73, Theorem 7]). Certainly $\mathfrak{F}$ is the group of permutations (see [73, Theorem 2]).

However, the connection between family $\mathfrak{F}$ and the convergent or divergent permutations is not noticeable directly, since it is located at the level of superpositions of convergent permutations, it means within families $\mathfrak{C}_2 = \mathfrak{DC} \circ \mathfrak{CD}$ and $\mathfrak{D}_2 = \mathfrak{CD} \circ \mathfrak{DC}$. One can prove that $\mathfrak{F} \subset \mathfrak{C}_2$ which implies also that $\mathfrak{DC} \circ \mathfrak{F} \circ \mathfrak{CD} \subset \mathfrak{C}_2$ (see [73, Conclusion 1]), and one can give the example of permutation $p \in \mathfrak{F} \setminus \mathfrak{D}_2$ (see [73, Example 1]). Inclusion $\mathfrak{F} \subset \mathfrak{C}_2$ results from the definitely more generally formulated theorem (see [73, Theorem 4]):

Theorem 6.1. *Let $p \in \mathfrak{F}$ and let $l(p)$ denote the smallest natural number such that: for each, sufficiently big $n \in \mathbb{N}$, there exists the partition $\mathfrak{N}(n)$ of set $\{k \in \mathbb{N} :\ k \geqslant n\}$ with $l(p) = \operatorname{card} \mathfrak{N}(n)$, such that $p\big|_N$ is the increasing map for each $N \in \mathfrak{N}(n)$. Then there exists the convergent permutation q satisfying conditions:*

$$qp \in \mathfrak{C},\ c_\infty(q) \leqslant 4l(p) + 1 \text{ and } c_\infty(qp) \leqslant 2l(p) + 1,$$

where

$$c_\infty(p) = \lim_{n\to\infty} \big(\sup\{c(p, I) :\ I \subset \mathbb{N} \text{ is the bounded interval such that } \min I \geqslant n\}\big),$$

whereas $c(p, I)$ denotes the number of mutually separated intervals creating the partition of set $p(I)$.

Coefficients, appearing here, can be successfully applied for more subtle formulation of many relations, also these ones previously given by me.

Moreover, I would like to emphasize that within many of discussed here families of permutations, the elements of family $\mathfrak{F}$ are spred out almost everywhere, in such sense that ([73, Theorems 3 and 6]):

- $\mathfrak{F} \cap A \neq \varnothing$ for each of four two-sided families $A \in \{\mathfrak{CC}, \mathfrak{CD}, \mathfrak{DC}, \mathfrak{DD}\}$, and even for $A = \mathfrak{I}$ (see Remark 3, [73]),
- $A \setminus \mathfrak{F} \neq \varnothing$ for each $A \in \{\mathfrak{CC}, \mathfrak{CD}, \mathfrak{D}(1) \cap \mathfrak{DC}, \mathfrak{D}(1) \cap \mathfrak{DD}, (\mathfrak{D}(2) \setminus \mathfrak{D}(1)) \cap \mathfrak{DC}, (\mathfrak{D}(2) \setminus \mathfrak{D}(1)) \cap \mathfrak{DD}, \mathfrak{D}(1) \cap \mathfrak{C}_2 \cap \mathfrak{D}_2\}$, where $\mathfrak{D}(2) := \{p \in \mathfrak{D} :\ \liminf_{n\to\infty} t(p, n) = 2\}$. However, I do not know whether $\mathfrak{F} \setminus A \neq \varnothing$ for each one of sets A mentioned above.

Family $\mathfrak{F}$ can be also used for strengthening the presentation of the Riemann Derangement Theorem (by the way this result comes from the Riemann's post-doctoral dissertation entitled "Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe", see [25, p. 232]). In paper [73] I have presented two of such theorems (Theorems 9 and 10) generalizing the Kronrod's results from paper [34] (see also [45, 21]). So, I have proven that for any conditionally convergent series $\sum a_n$ of real terms and closed interval $I \subset \mathbb{R} \cup \{\pm\infty\}$ there exist permutations $p \in \mathfrak{F}$ and $q \in \mathfrak{P} \setminus \mathfrak{F}$, such that the set of limit points of every series $\sum a_{p(n)}$ and $\sum a_{q(n)}$ is equal to I. In paper [67] similar result can be found but for permutation $p \in \mathfrak{DD}$, and in case when $\sum a_n \in I$ or when $I = [\alpha, \infty]$ or $I = [-\infty, \beta]$, $\alpha < \infty$, $\beta > -\infty$, $\alpha, \beta \in \mathbb{R} \cup \{\pm\infty\}$, for permutation $q \in \mathfrak{DC}$ with additional condition $c(q^{-1}) \leqslant 5$ (which may be strengthened to $c(q^{-1}) \leqslant 3$, when $\sum a_n \in I$). Definition of number $c(\varphi)$, $\varphi \in \mathfrak{C}$, is given on page 182. Let us noticed that $c_\infty(\varphi) \leqslant c(\varphi)$, $\varphi \in \mathfrak{C}$.

We note also that J.H. Smith in [55] considers the selection of permutations p in the Riemann Derangement Theorem with the predetermined decomposition of p into cycles.

7. Sequences conglomerating intervals and growing the numbers of intervals

Let $p \in \mathfrak{P}$ and let $\sigma(t(p,\circ))$ denote the set of limit points of sequence $\{t(p,n)\}_{n=1}^{\infty}$ with regard to topology of the 2-points compactification of set $\mathbb{R}$ with the standard topology, in other words $\sigma(t(p,\circ))$ is derivative of this sequence. Then the following relations hold true.

Lemma 7.1.

(i) $\{t(p,n)\}_{n=1}^{\infty} \subset \mathbb{N}$ *and* $t(p,n+1) - t(p,n) \in \{-1,0,1\}$, *for every* $n \in \mathbb{N}$,

(ii) *set* $\sigma(t(p,\circ))$ *is the closed interval* $\subset \mathbb{N} \cup \{\infty\}$, *more precisely we have* $\sigma(t(p,\circ)) = [\liminf_{n\to\infty} t(p,n), \limsup_{n\to\infty} t(p,n)]$,

(iii) *if* $\operatorname{card}(\sigma(t(p,\circ))) = 1$ *then either* p *is the almost identity permutation on* $\mathbb{N}$ *or* $\sigma(t(p,\circ)) = \{\infty\}$, *that is* $\lim_{n\to\infty} t(p,n) = \infty$,

(iv) $1 \in \sigma(t(p,\circ))$ *if and only if* $\liminf_{n\to\infty} t(p,n) = 1$, *if we assume additionally that* p *is divergent, then* $1 \in \sigma(t(p,\circ))$ *if and only if* $p \in \mathfrak{D}(1)$,

(v) *if there exists an increasing sequence* $\{x_n\}_{n=1}^{\infty}$ *of natural numbers such that*

$$\limsup_{n\to\infty}(x_{n+1} - x_n) < \infty \quad \text{and} \quad \lim_{n\to\infty}(p(x_{n+1}) - p(x_n)) = \infty,$$

then

$$\lim_{n\to\infty} t(p,n) = \lim_{n\to\infty} t(p^{-1},n) = \infty.$$

(vi) *If* A *and* B *are any closed intervals of the form* $[k,l]$, $[k,\infty]$ *or* $[\infty]$, *where* $k,l \in \mathbb{N}$, $k < l$, *then there exists permutation* $p \in \mathfrak{P}$ *such that* $\sigma(t(p,\circ)) = A$ *and* $\sigma(t(p^{-1},\circ)) = B$.

Remark 7.2. Property (v) plus the combinatoric characterization of permutation $p \in \mathfrak{S}_0$ enable to give easily the previously cited construction of permutation from paper by Hu and Wang [29]. The matter is just the idea of this construction.

Let us now assign to permutation p two auxiliary sets ($t(p,0) := 0$):

$$\mathfrak{U}(p) := \{u \in \mathbb{N} : \ t(p,u) - t(p,u-1) = 1\}$$

and

$$\mathfrak{V}(p) := \{v \in \mathbb{N} : \ t(p,v) - t(p,v-1) = -1\}.$$

Lemma 7.3 (see [66])**.**

(i) *If* $p \in \mathfrak{D}$, *then the sets* $\mathfrak{U}(p)$ *and* $\mathfrak{V}(p)$ *are infinite. More precisely, if at least one of these sets is finite then the other one is finite as well, whereas* p *is then the almost identity permutation.*

(ii) *If we denote by* $I_{i,n}^{(p)}$, $i = 1,2,\dots,t(p,n)$ *the sequence of msi creating the partition of set* $p([1,n])$ *for each* $n \in \mathbb{N}$ *then*

$$\mathrm{card}(p^{-1}(I_{i,n}^{(p)}) \cap \mathfrak{U}(p)) - \mathrm{card}(p^{-1}(I_{i,n}^{(p)}) \cap \mathfrak{V}(p)) = 1$$

for each $i, n \in \mathbb{N}$, $i \leqslant t(p,n)$.

(iii) $\mathrm{card}([1,n] \cap \mathfrak{U}(p)) - \mathrm{card}([1,n] \cap \mathfrak{V}(p)) = t(p,n)$ *for each* $n \in \mathbb{N}$.

Let us assume additionally that permutation $p \in \mathfrak{P}$ is not the almost identity permutation (in particular, it can be the divergent permutation). Then the sets $\mathfrak{U}(p)$ and $\mathfrak{V}(p)$ are infinite and the increasing sequences of all elements of sets $p(\mathfrak{U}(p))$ and $p(\mathfrak{V}(p))$ will be denoted by $\{u_n(p)\}_{n=1}^{\infty}$ and $\{v_n(p)\}_{n=1}^{\infty}$, respectively, and called the sequences growing the numbers of intervals and conglomerating the intervals (adequately to their properties).

Lemma 7.4 (see [66]). *We have*

$$u_n(p) < v_n(p) < u_{n+1}(p) \qquad \textit{and} \qquad p^{-1}(u_n(p)) < p^{-1}(v_n(p))$$

for each $n \in \mathbb{N}$.

I have also proven that (see [66]):

Theorem 7.5. *If* $\{x_n\}_{n=1}^{\infty}$ *is the increasing sequence of natural numbers then there exist permutations* $p \in \mathfrak{DC}$ *and* $q \in \mathfrak{DD}$ *satisfying conditions*

$$\lim_{n\to\infty} t(\varphi, n) = \infty$$

and

$$\varphi(\mathfrak{U}(\varphi)) = \{x_{2n-1}\}_{n=1}^{\infty} \quad \text{and} \quad \varphi(\mathfrak{V}(\varphi)) = \{x_{2n}\}_{n=1}^{\infty}$$

for each $\varphi \in \{p, q\}$.

Remark 7.6. The above theorem is the strengthened version of Theorem 5.3 from paper [66]. The proof is given in paper [75].

Remark 7.7. By using properties (ii) and (iii) from Lemma 7.3 we can prove the previously given characterizations of divergent permutations connected with the idea of splicing the sequences. For example, let $p \in \mathfrak{D}$ and $k \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that $t(p,n) > k$. We select an element x_i from each interval $I_{i,n}^{(p)}$ for every $i = 1, 2, \ldots, t(p,n)$. Assuming that $I_{i,n}^{(p)} < I_{i+1,n}^{(p)}$, $i = 1, 2, \ldots, t(p,n) - 1$, we obtain that the sequence $\{x_i\}_{i=1}^{t(p,n)}$ is increasing. We also select elements $y_i^* \in p^{-1}((\max I_{i,n}^{(p)}, \min I_{i+1,n}^{(p)}))$, $i = 1, 2, \ldots, t(p,n)-1$ and $y_{t(p,n)}^* \in p^{-1}((\max I_{t(p,n),n}^{(p)}, \infty))$. Let $\{x_i\}_{i=1+t(p,n)}^{2t(p,n)}$ be an increasing sequence composed from elements y_i^*, $i = 1, 2, \ldots, t(p,n)$. Certainly the sequence $\{x_i\}_{i=1}^{2t(p,n)}$ is spliced by p. Application of the Erdős-Szekeres Theorem[7] [80] enables to assume additionally that sequences $\{p(x_i)\}_{i=1}^{t(p,n)}$ and $\{p(x_i)\}_{i=1+t(p,n)}^{2t(p,n)}$ are both monotonic.

[7] **Remark.** In paper [80] several essential supplements for the Erdős-Szekeres Theorem are also presented. For example, it has been revealed that the existence of increasing or decreasing subsequence composed from three successive elements of investigated sequence, it means the monotonic subsequence of the form $\{a_k, a_{k+1}, a_{k+2}\}$, is of great importance for discussing problems of that kind. A part of discussed there problems can be described in the language of theory of "pattern avoiding permutations" (see [7, 11]).

Remark 7.8. Sets $\mathfrak{U}(p)$ and $\mathfrak{V}(p)$ can be also used for constructing the convergent real series $\sum a_n$, the p-rearrangements of which possess the set of limit points determined in advance. For instance, in paper [66, Theorem 4.4] the following result is proven:

Theorem 7.9. *Let $\{p_i\}_{i=1}^{\infty} \subset \mathfrak{D}$ and $\lim_{n\to\infty} t(p_i, n) = \infty$ for every $i \in \mathbb{N}$. If permutations p_i, $i \in \mathbb{N}$, possess the identical sequences growing the numbers of intervals and conglomerating the intervals, then for each $\alpha \in \mathbb{R}$ there exist the series $\sum a_n$ and $\sum b_n$ of real numbers satisfying conditions:*

(1) $\sum a_n = \sum b_n = \alpha$,

(2) $\sum_{n=1}^{\infty} a_{p_i(n)} = \infty$, for each $i = 1, 2, \ldots$,

(3) set of limit points of every series $\sum_{n=1}^{\infty} b_{p_i(n)}$, $i = 1, 2, \ldots$ is equal to $[\alpha, \infty]$.

8. Relations between the convergence classes of permutations on $\mathbb{N}$

Convergence class of permutation $p \in \mathfrak{P}$ is the family of all the convergent series $\sum a_n$ of real terms, such that the p-rearranged series $\sum a_{p(n)}$ is convergent as well. Convergence class of given permutation $p \in \mathfrak{P}$ will be denoted by symbol $\sum(p)$. Certainly if $p \in \mathfrak{C}$ then $\sum(p)$ is the family of all convergent real series. Therefore, from theoretical point of view, only the convergent classes of divergent permutations can be interesting.

I have considered many various problems concerning this subject. For example, in paper [79] I have introduced the idea of strongly and weakly divergent permutations $p \in \mathfrak{D}$ in dependence on that whether they fulfil the condition, respectively:

$$\lim_{n\to\infty} t(p, n) = \infty, \quad \text{or} \quad \liminf_{n\to\infty} t(p, n) < \infty.$$

In other words, permutation $p \in \mathfrak{D}$ is strongly divergent if it rearranges some convergent real series into the series divergent to ∞, whereas $p \in \mathfrak{D}$ is weakly divergent if it belongs to some $\mathfrak{D}(k)$, $k \in \mathbb{N}$, where $\mathfrak{D}(k) := \{p \in \mathfrak{D} : \liminf_{n\to\infty} t(p, n) = k\}$, it means if for each convergent real series $\sum a_n$ the sum of series $\sum a_n$ in the set of limit points of $\sum a_{p(n)}$ can be found. So, $p \in \mathfrak{D}$ is weakly divergent if and only if $p \in \bigcup_{k\in\mathbb{N}} \mathfrak{D}(k)$. Family $\bigcup_{k\in\mathbb{N}} \mathfrak{D}(k)$ has been introduced by Kronrod [34], however he did not distinguish families $\mathfrak{D}(k)$, $k \in \mathbb{N}$, separately. Let us also notice that the weakly divergent permutations are simultaneously the sum preserving permutations since $\bigcup_{k\in\mathbb{N}} \mathfrak{D}(k) \subset \mathfrak{S}_0$.[8]

[8] I prove in [63, Section 2] that if $p \in \mathfrak{D}(k)$ for some $k \in \mathbb{N}$ and $\sum a_n$ is a conditionally convergent series such that the set of limit points of series $\sum a_{p(n)}$ is equal to $[\alpha, \beta] \subset \mathbb{R}$, then $k(\alpha - \beta) + \beta \leqslant \sum a_n \leqslant k(\beta - \alpha) + \alpha$. Hence it follows that the intervals $[\alpha, \beta]$ with $k(\beta - \alpha) + \alpha < \sum a_n$ or $k(\alpha - \beta) + \beta > \sum a_n$ cannot be the sets of limit points of series $\sum a_{p(n)}$ for any $p \in \mathfrak{D}(k)$. We note also that Kronrod [34, Theorems 6, 6a, 7] shows that if p^{-1} is a weakly divergent permutation,

Let me begin the problem concerning mutual relations between the convergence classes of strongly and weakly divergent permutations with the following fact. There exist the strongly divergent permutation p and the weakly divergent permutations q_1 and q_2 fulfilling relations

$$\sum(q_1) = \sum(p) \quad \text{and} \quad \sum(q_2) \subset \sum(p),$$

– see Example 2 in paper [79]. I have proven (see [79, Theorem 1]) that for each strongly divergent permutation p there exists a weakly divergent permutation q such that $\sum(q) \subseteq \sum(p)$ and, under some additional assumptions about permutation p, there exist the weakly divergent permutations q_1 and q_2 fulfilling the relations as in the example given above.

In paper [79] there is also proven that for each strongly divergent permutation $p \in \mathfrak{DC}$ there exists a weakly divergent permutation q, such that $\sum(p) = \sum(q)$. Moreover, an example of strongly divergent permutation $p \in \mathfrak{DC}$ is given. This example essentially completes the example from paper [29], i.e. the strongly divergent permutation, the inverse permutation of which is strongly divergent as well.

The next problem concerning the convergence classes is connected with a question about the possibility of restricting or expanding the convergence class of the given divergent permutation – it appears that it is always possible and not uniquely at all. I will precede the discussion by an example showing the subtleness of such operations. In paper [81] I have given the example of permutations $p, q \in \mathfrak{D}(1)$ such that if $\sum(p) \cup \sum(q) \subseteq \sum(\sigma)$ then $\sigma \in \mathfrak{C}$. Simultaneously, set $\sum(p) \cap \sum(q)$ is the convergence class of some permutation from $\mathfrak{D}(1)$. Analyzing the construction of these permutations p, q, I have noticed that one of the following three conditions: $p, q \in \mathfrak{DC}$ or $p \in \mathfrak{DC}$ and $q \in \mathfrak{DD}$ or $p, q \in \mathfrak{DD}$ can be additionally assumed here.

We consider now the thread of restrictions – expansions of the convergence class of given permutation $p \in \mathfrak{D}$. In paper [81] – fifth section, there is introduced a family $\Omega \subset \mathfrak{D}$,[9] $\mathfrak{DC} \subset \Omega$ and $\mathfrak{DD} \cap \Omega \neq \varnothing$ such that for any permutation $p \in \Omega$ there exists a subset $\Omega(p) \subset \Omega$, $\operatorname{card}(\Omega(p)) = \mathfrak{c}$ satisfying two basic conditions:

$$\sum(p) \subset \sum(\varphi)$$

for each $\varphi \in \Omega(p)$ and any two permutations $\varphi, \psi \in \Omega(p)$ are incomparable, which will hereafter mean that

$$\sum(\varphi) \setminus \sum(\psi) \neq \varnothing \text{ and } \sum(\psi) \setminus \sum(\varphi) \neq \varnothing.$$

In particular, if $p \in \mathfrak{DC}$ then also $\Omega(p) \subset \mathfrak{DC}$ and one can assume that $c(\varphi^{-1}) \leqslant 4c(p^{-1}) + 1$ for each $\varphi \in \Omega(p)$, where

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$\sum a_n = \infty$ (or $-\infty$) and series $\sum a_{p(n)}$ is convergent to some point from $\mathbb{R} \cup \{\pm\infty\}$ then we have $|\sum a_{p(n)}| = \infty$.

[9] Permutation $p \in \mathfrak{P}$ belongs to Ω if there exists an increasing sequence $\{I_n(p)\}_{n=1}^{\infty}$ of intervals satisfying three conditions: sequence $\{p^{-1}(I_n(p))\}_{n=1}^{\infty}$ is increasing as well, there exists constant $k = k(p) \in \mathbb{N}$ such that each of sets $p^{-1}(I_n(p))$ is a union of at most k msi and, moreover, one can indicate sequence $\{J_n\}_{n=1}^{\infty}$ of intervals such that $J_n \subset p^{-1}(I_n(p))$ and $\lim\limits_{n\to\infty} c(p, J_n) = \infty$.

$$c(q) := \sup\{c(q,I) : I \subset \mathbb{N} \text{ is the bounded interval}\}$$

for any convergent permutation q. Study of the proof of these relations given in paper [81] implies that one can always suppose that only weakly divergent permutations belong to $\Omega(p)$.

I have announced in paper [69] one more result (Theorem 6.1): if $p \in \mathfrak{DC}$ then there exists family $\Omega(p) \subset \mathfrak{DC}$, $\mathrm{card}(\Omega(p)) = \mathfrak{c}$, incomparable internally (which will denote, by virtue of definition, that any two different permutations belonging to this family are incomparable) and such that (the proof is presented in [78]):

$$\sum(p) \subset \bigcap_{\omega \in \Omega(p)} \sum(\omega).$$

Remark 8.1. There exist many unexpected relations (including the inclusions) connected with the one-sided divergent permutations. Even a single thing, that if $p, q \in \mathfrak{DC}$ then

$$\sum(pq) \subset \sum(p)$$

which means in consequence that, for example

$$\sum(p^{n+1}) \subset \sum(p^n)$$

for any $n \in \mathbb{N}$ and $p \in \mathfrak{DC}$ and, additionally, that in family $\mathfrak{DC}$ does not exist a permutation with minimal convergence class, with regard to inclusion between the convergence classes. Similarly, from the fact that $\mathfrak{DC} \circ \mathfrak{DC} = \mathfrak{DC}$ (see [69, Theorem 2.2]) it results that for each permutation $p \in \mathfrak{DC}$ there exists permutation $q \in \mathfrak{DC}$ such that $\sum(p) \subset \sum(q)$, in consequence, in family $\mathfrak{DC}$ does not exist a permutation with maximal convergence class, with regard to inclusion between the convergence classes. However, one can find some other, unsolved yet, problems concerning the existence of effects of the so called countable maximality and, respectively, countable minimality.[10] One can also formulate these problems individually, within each of families $\mathfrak{DC}$, $\mathfrak{D}$, $\mathfrak{DD}$ with regard to the relation of inclusion between the convergence classes of given permutations.

[10] Let $<$ be a binary and transitive relation on the infinite set X. We say that X possesses the effect of countable maximality if the following two conditions are satisfied

a) in set X there are no elements maximal with regard to relation $<$;
b) for each $x_0 \in X$ there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that

$$x_0 < x_1 < x_2 < \dots$$

and there does not exist an upper bound of $\{x_n\}_{n=1}^{\infty}$, i.e. there does not exist $y \in X$ such that

$$x_n < y \qquad \text{for every } \; n \in \mathbb{N}.$$

Similarly we define the effect of countable minimality. For example, in set of rational numbers taken from the given nonempty open interval included in $\mathbb{R}$ with relation “less than” $<$ both effects occur: countable minimality as well as countable maximality, whereas in family of countable subsets of $\mathbb{R}$ with the inclusion relation the effect of countable maximality does not appear.

I have proven in paper [69] one more important theorem:

Theorem 8.2. *Let $p \in \mathfrak{DC}$, $\Omega_1 \subset \mathfrak{D}$ and $\Omega_2 \subset \mathfrak{DC}$. If both sets Ω_1 and Ω_2 are nonempty and countable then the relations hold*

$$\sum(p) \setminus \bigcup_{\omega\in\Omega_1} \sum(p\omega) \neq \varnothing \qquad \textit{and} \qquad \bigcup_{\omega\in\Omega_2} \sum(p\omega) \subset \sum(p).$$

Additionally, I have proven the following fact [78]: let $\Gamma \subset \mathfrak{D}$ be the countable infinite family. Then the permutations $p \in \mathfrak{DC}$ and $q \in \mathfrak{DD}$ exist such that

$$\sum(\varphi) \setminus \bigcup_{\gamma\in\Gamma} \sum(\gamma) \neq \varnothing \quad \text{and} \quad \bigcap_{\gamma\in\Gamma} \sum(\gamma) \setminus \sum(\varphi) \neq \varnothing$$

for every $\varphi \in \{p, q\}$.

I have also investigated the following problem: let $p \in \mathfrak{P}$ and $Q \subset \mathfrak{P}$, $Q \neq \varnothing$. Let us assume that for each $q \in \mathbb{Q}$ we have $\sum(p) \setminus \sum(q) \neq \varnothing$. I have proven ([69, Theorem 4.1]) that if Q is a finite set, then also $\sum(p) \setminus \bigcup\limits_{q\in Q} \sum(q) \neq \varnothing$. As yet, it is unknown whether the relation holds also when Q is a countable set (I have only received the partially positive result – Theorem 4.3 in the same paper, for example when additionally $p \in \mathfrak{DC}$).

In paper [81] I have also proven that for each permutation $p \in \mathfrak{DD}$ there exists family $\Omega(p) \subset \mathfrak{DD}$, $\operatorname{card}\Omega(p) = \mathfrak{c}$ satisfying two conditions, similarly like in case of family $\Omega(p) \subset \Omega$ discussed above (see Theorem 4.5 and Remark 4.6 in [81]). Analyzing proof of this theorem I have noticed that if we assume that p is the weakly divergent permutation, then the appropriate family $\Omega(p)$ is created from the weakly divergent permutations as well. This assertion remains true also if p is the strongly divergent permutation (in proof of this case some additional assumptions must be used). One should also mention that the received here family $\Omega(p)$ additionally fulfils condition (for each $q \in \Omega(p)$):

$$\sum(q) \setminus \Bigg(\sum(p) \cup \bigcup_{\substack{\varphi\in\Omega(p)\\ \varphi\neq q}} \sum(\varphi) \Bigg) \neq \varnothing.$$

In paper [69, Theorem 5.1] I have proven the theorem dual to the above theorem concerning the restriction of convergence class of the given divergent permutation p. I have shown that for each permutation $p \in \mathfrak{D}$ there exists family $\Phi(p) \subset \mathfrak{DC}$, $\operatorname{card}\Phi(p) = \mathfrak{c}$, such that

$$\bigcup_{\varphi\in\Phi(p)} \sum(p\varphi) \subset \sum(p)$$

and each of families $\Phi(p)$ and $\{p\} \circ \Phi(p)$ is finitely internally incomparable.[11]

[11] We say that set $\Phi \subset \mathfrak{P}$ is finitely internally incomparable if for each finite set $F \subset \Phi$, $F \neq \varnothing$ the following relations hold

$$\Big(\bigcap_{f\in F} \sum(f) \Big) \setminus \Big(\bigcup_{\varphi\in\Phi\setminus F} \sum(\varphi) \Big) \neq \varnothing \quad \text{and} \quad \Big(\bigcap_{\varphi\in\Phi\setminus F} \sum(\varphi) \Big) \setminus \Big(\bigcup_{f\in F} \sum(f) \Big) \neq \varnothing.$$

I have also given an additional condition which should be satisfied by permutation p so that it would be sufficient to construct family $\Psi(p) \subset \mathfrak{DD}$ with $\operatorname{card}\Psi(p) = \mathfrak{c}$ fulfilling the same conditions as family $\Phi(p)$. Slight modification of these conditions would ensure the weak divergence of p and permutations belonging to $\Psi(p) \cup (\{p\} \circ \Psi(p))$.

Remark 8.3. We say that nonempty family $\mathfrak{T} \subset \mathfrak{D}$ is totally internally incomparable if for every nonempty $S \subset \mathfrak{T}$ the relation holds

$$\bigcap_{s \in S} \sum(s) \setminus \bigcup_{t \in \mathfrak{T} \setminus S} \sum(t) \neq \varnothing.$$

I have also proven that if the nonempty finite or countable family $\mathfrak{T} \subset \mathfrak{D}$ is totally internally incomparable, then there exist permutations $p \in \mathfrak{DD} \setminus \mathfrak{T}$ and $q \in \mathfrak{DC} \setminus \mathfrak{T}$ such that the sets $\{p\} \cup \mathfrak{T}$ and $\{q\} \cup \mathfrak{T}$ are totally internally incomparable as well. It means that there do not exist the maximal, with regard to inclusion, countable sets of permutations totally internally incomparable (it will be the subject of separate publication [78]). Problem is as follows: does it exist the family of countable sets $\mathfrak{T} \subset \mathfrak{D}$ totally internally incomparable which with respect to the inclusion relation between convergence classes possesses the effect of countable maximality?

Problem of the convergence classes is completed by the concept of Levi classes discussed in papers [23, 57].

9. On some equivalence relation and topologies defined on family $\mathfrak{P}$

Let me introduce now some equivalence relation on $\mathfrak{P}$, equivalence classes of which strictly fit in all the previously distinguished subfamilies $\mathfrak{P}$. First we assign to permutations $p, q \in \mathfrak{P}$ the quasi-metric

$$d(p, q) := \limsup_{n \to \infty} \{|p(n) - q(n)|\}.$$

Next we define the mentioned equivalence relation ϱ_d on $\mathfrak{P}$ by condition

$$p \varrho_d q \Leftrightarrow d(p, q) < \infty.$$

The following important fact, concerning the equivalence class $[p]_{\varrho_d}$ under ϱ_d of any $p \in \mathfrak{P}$, holds:

Theorem 9.1. *If $p \in \mathfrak{A}$, then $[p]_{\varrho_d} \subset \mathfrak{A}$, for each $\mathfrak{A} \in \{\mathfrak{CC}, \mathfrak{C}, \mathfrak{CD}, \mathfrak{DC}, \mathfrak{S}, \mathfrak{DD}, \mathfrak{I}\}$.*

Presented theorems and ideas are well completed by considerations from paper [12] (see also [28, 36, 50]).

The nonmetrizable topologies on $\mathfrak{P}$ were defined by Steven G. Krantz (well known authority on the several complex variables) and Jeffrey D. Mcneal in [33, Section 4].

In contrast to some standard metric topologies on $\mathfrak{P}$ (for example, Baire's metric and Fréchet's metric, however we note that these both metrics are equivalent on $\mathfrak{P}$) were discussed by many authors:

(1) R.P. Agnew in [1],
(2) F.A. Talaljan in [58],
(3) L. Drewnowski in [19],
(4) R.G. Bilyeu, R.R. Kallman and P.W. Lewis in [4] – this paper is strongly connected with the subject of presented monograph since it solves some problem, introduced by Kac and Zygmund, about the category of the set of rearrangements making the given Fourier series divergent almost everywhere. It is related to Kolmogorov's problem "that there exists an L^2– Fourier series such that some rearrangement of it diverges almost everywhere", proved by Zahorski. Prof. Władysław Wilczyński writes about this as well in his paper "Zygmunt Zahorski and contemporary real analysis" included in this monograph,
(5) H. Miller and E. Özturk in [36],
(6) M. Bhaskara Rao, K.P.S. Bashkara Rao and B.V. Rao in [3], where the category aspect of family $\mathfrak{C}$ is especially distinguished,
(7) and at last by J. Červeňanský and T. Šalát in [10] and T. Šalát in [48] where the authors applied the concept of the porosity of sets for making more precise and completing the results on $\mathfrak{P}$ for Fréchet's metric proved by the other authors. See also paper [17] by M. Dindoš, I. Martišovitš and T. Šalát.

10. Outline of the history of research on the convergent and divergent permutations

1890 – Emil Borel in paper [8] gave the first description of convergent permutations (of the analytical-combinatoric nature). Today we could say that this description does not satisfy the expectations, however we owe to this paper the alternative definition of convergent permutations – the borelian permutations. From the private correspondence of Z. Sawoń (1989) I got to know that problems concerning the borelian permutations were investigated by Professor S. Mazur and participants of his workshop on the Warsaw University in the 1960's. However the elaborated results have never been published.

1946 – F.W. Levi in paper [35] gave the first fully combinatoric description of convergent permutations and considered the problems continued afterwards by Stout [57], Tusnády [59] and Gerencser [23] (resulted, among others, in introducing the so called Levi classes. For the contrast, there is distinguished in literature the Lévy group $G \subset \mathfrak{P}$ which is tightly linked to the notion of asymptotic density and was discussed, for example, by Sleziak and Ziman [54], Blümlinger [5] and Obata [41, 42, 6], Nathanson and Parikh [40].) Stout [57] proved also that $p \in \mathfrak{P}$ rearranges every alternating real series $\sum a_n$ onto convergent series $\sum a_{p(n)}$ if and only if p possesses finite balans, i.e. if the sequence $\left\{ \sum_{k=1}^{n} (-1)^{p(k)} \right\}_{n=1}^{\infty}$ is bounded.

1951 – as a matter of fact probably already in the 1940's, N. Bourbaki [9] (more precisely, some of his collaborators - however I do not know which one exactly) presented the next combinatoric characterization of convergent permutations: permutation $p \in \mathfrak{P}$ is convergent if and only if there exists constant $c = c(p) \in \mathbb{N}$ such that the set $p([1, n] \cap \mathbb{N})$ is a union of at most c msi for every $n \in \mathbb{N}$. This characterization is at present the most common characterization of convergent permutations and it is most often attributed to R.P. Agnew!

1955 – R.P. Agnew's paper [2] appeared in which the above given characterization of convergent permutations was presented. One should suppose that Agnew was not aware of the fact that his characterization has been published earlier in Bourbaki's monograph [9].

1946 – Russian mathematician A.S. Kronrod published very important paper [34] (presented by Mienszov). Let us emphasize the fact that because of the time of this publication (right after the Second World War) the paper was completely unknown by mathematicians from the other side of the Iron Curtain (and it is actually happening till the present time). Kronrod introduced and discussed for the first time the families $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{CC}$, $\mathfrak{DC}$, $\mathfrak{CD}$, $\mathfrak{DD}$, $\mathfrak{P} \setminus \mathfrak{S}$ (Kronrod used different terminology and notations than I am using). Important is that Kronrod gave the combinatoric characterization of divergent permutations which is dual to the Bourbaki-Agnew's combinatoric characterization of convergent permutations. Kronrod gave also the example of permutation $p \in \mathfrak{DD} \cap \mathfrak{S}$ and he proved the following relations: $\mathfrak{C} \circ \mathfrak{C} \subset \mathfrak{C}$, $\mathfrak{C}^{-1} \circ \mathfrak{C}^{-1} \subset \mathfrak{C}^{-1}$ and $\mathfrak{C} \circ \mathfrak{S} = \mathfrak{S} \circ \mathfrak{C}^{-1} = \mathfrak{P}$. Let me notice that we obtain from there the inclusion $\mathfrak{P} \setminus \mathfrak{S} \subset \mathfrak{S} \circ \mathfrak{DC}$ which leads to the conclusion that for every conditionally convergent real series $\sum a_n$ there exists a permutation $p \in \mathfrak{DC}$ such that the series $\sum a_{p(n)}$ is divergent. Kronrod investigated as well the generalizations of the Riemann Derangement Theorem and the Steinitz Theorem (restricted only to the complex series, see monograph [32] and paper [24]). Main point of these generalizations consisted in selecting the respective permutation from the family $\mathfrak{C}_2$ $(= \mathfrak{DC} \circ \mathfrak{CD} = \mathfrak{C}^{-1} \circ \mathfrak{C})$. See also papers motivated by these Kronrod's results [45, 66, 67, 63].

I would like to mention that the Kronrod's paper was introduced to me in the 90's by means of the Jasek's survey paper [30], very important paper in historical meaning for the theory of series. Let as recall that Jasek initiated the new topic in research on the rearrangements of scalar and complex series, which concerned the characterizations of permutations $p \in \mathfrak{P}$ such that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (a_{p(k)} - a_k) = 0$$

for every real series $\sum a_k$ with $a_k \to 0$ for $k \to \infty$. Solutions of this problem can be found in papers [13, 14, 15, 16] of P.H. Diananda. More information concerning the mathematical creations of B. Jasek is given in [27].

1977 – Pleasants [43, 44] showed that $\mathcal{G} \neq \mathfrak{P}$. From this it results immediately that $\mathfrak{S} \neq \mathcal{G}$ because $\mathfrak{S}$ is the algebraically big subset of $\mathfrak{P}$. Moreover, in [77] we proved the following fact.

If we fix the increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that $n_0 = 1$ and

$$\limsup(n_{k+1} - n_k) = \infty, \tag{2}$$

then the family $G = G\left(\{n_k\}_{k=0}^{\infty}\right)$ of permutations $p \in \mathfrak{P}$, such that permutation p maps each interval $[n_{k-1}, n_k)$, $k = 1, 2, ...,$ onto itself, satisfies the following conditions:

- it is a subgroup of $\mathfrak{P}$,
- $G \cap \mathfrak{D} = G \cap \mathfrak{D}(1)$ (the remaining elements from G belong to $\mathfrak{C}$),
- $G \subset \mathfrak{S}$,
- $G \cap \mathfrak{X} \neq \emptyset$ for each $\mathfrak{X} \in \{\mathfrak{CC}, \mathfrak{CD}, \mathfrak{DC}, \mathfrak{DD}\}$,
- $G \setminus \mathcal{G} \neq \emptyset$ (essential generalization of Pleasants's result),
- if $\lim\limits_{k\to\infty}(n_{k+1} - n_k) = \infty$ and $\lim\limits_{k\to\infty} \frac{n_{k+1}}{n_k} = 1$ then G is a subgroup of the Lévy group (see [41]).

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